

New Periodic Solutions for Second Order Hamiltonian Systems with Local Lipschitz Potentials

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(Dedicated to the Memory of Professor Shi Shuzhong)

Abstract Firstly, we generalize the classical Palais-Smale-Cerami condition for C^1 functional to the local Lipschitz case, then generalize the famous Benci-Rabinowitz's and Rabinowitz's Saddle Point Theorems with classical Cerami-Palais-Smale condition to the local Lipschitz functional, then we apply these Theorems to study the existence of new periodic solutions for second order Hamiltonian systems with local Lipschitz potentials which are weaker than Rabinowitz's original conditions. The key point of our proof is proving Cerami-Palais-Smale condition for local Lipschitz case, which is difficult since no smooth and symmetry for the potential.

Key Words: Second order Hamiltonian systems, Cerami-Palais-Smale condition for local Lipschitz functional, Periodic solutions, Saddle Point Theorems.

2000 Mathematical Subject Classification: 34C15, 34C25, 58F.

1. Introduction

In the critical point theory, the compactness condition is a key for proving the existence of critical points for some functionals. In 1964, R. Palais and S. Smale [13] introduced the famous $(PS)_c$ condition:

Definition 1.1 Let X is a Banach space, $f \in C^1(X, R)$, if $\{x_n\} \subset X$ s.t.

$$\begin{aligned} f(x_n) &\rightarrow c, \\ f'(x_n) &\rightarrow 0, \end{aligned}$$

and $\{x_n\}$ has a strongly convergent subsequence, then we say f satisfies $(PS)_c$ condition.

In 1978, Cerami [4] presented a weaker compactness condition than the above classical $(PS)_c$ condition:

Definition 1.2 Let X be a Banach space, Φ be defined on X is Gateaux-differentiable, if the sequence $\{x_n\} \subset X$ such that

$$\Phi(x_n) \rightarrow c,$$

$$(1 + \|x_n\|)\|\Phi'(x_n)\| \rightarrow 0,$$

then $\{x_n\}$ has a strongly convergent subsequence in X . Then we call f satisfies $(CPS)_c$ condition in X .

For the functional $f(x)$ in locally Lipschitz functional space $C^{1-0}(X, R)$, Clarke [6] define the generalized gradient $\partial f(x)$ which is the subset of X^* defined by

$$\partial f(x) = \{x^* \in X^* | \langle x^*, v \rangle \leq f^0(x, v), \forall v \in X\},$$

where

$$f^0(x, v) = \lim_{y \rightarrow x, \lambda \downarrow 0} \sup \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

In 1981, K.C.Chang[5] introduced the (PS) condition for locally Lipschitz function:

Definition 1.3 Let X is a Banach space, $f \in C^{1-0}(X, R)$, if $\{x_n\} \subset X$ s.t. $f(x_n)$ is bounded and

$$\min_{x^* \in \partial f(x_n)} \|x^*\| \rightarrow 0,$$

and $\{x_n\}$ has a strongly convergent subsequence, then we say f satisfies (PSC) condition.

if $\{x_n\} \subset X$ s.t. $f(x_n) \rightarrow c$ and

$$\min_{x^* \in \partial f(x_n)} \|x^*\| \rightarrow 0,$$

and $\{x_n\}$ has a strongly convergent subsequence, then we say f satisfies $(PSC)_c$ condition.

Ekeland [8], Ghoussoub-Preiss[9] used Ekeland's variational principle to prove

Lemma 1.1 Let X be a Banach space, suppose that Φ defined on X is Gateaux-differentiable and lower semi-continuous and bounded from below. Then there is a sequence $\{x_n\}$ such that

$$\begin{aligned} \Phi(x_n) &\rightarrow \inf \Phi \\ (1 + \|x_n\|)\|\Phi'(x_n)\| &\rightarrow 0. \end{aligned}$$

Motivated by the above Definitions and Lemma, we introduce the following (CPS)-type condition for the locally Lipschitz functional:

Definition 1.4 Let X is a Banach space, $f \in C^{1-0}(X, R)$, we say f satisfies $(CPSC)_c$ condition if $\{x_n\} \subset X$ s.t.

$$\begin{aligned} f(x_n) &\rightarrow c, \\ (1 + \|x_n\|)\min_{x^* \in \partial f(x_n)} \|x^*\| &\rightarrow 0, \end{aligned}$$

then $\{x_n\}$ has a strongly convergent subsequence.

K.C.Chang[5] and Shi S.Z.[16] use the (*PSC*) condition for the local Lipschitz functional to generalize the classical Mountain Pass Lemma[2] and general minimax Theorems[12]. Here we can generalize the classical Benci-Rabinowitz's and Rabinowitz's Saddle Point Theorems to the local Lipschitz functional cases with the Cerami-Palais-Smale-Chang-type conditions:

Theorem1.1 Let X be a Banach space, $f \in C^{1-0}(X, R)$. Let $X = X_1 \oplus X_2$, $\dim X_1 < +\infty$, X_2 is closed in X . Let

$$\begin{aligned} B_a &= \{x \in X \mid \|x\| \leq a\}, \\ S &= \partial B_\rho \cap X_2, \rho > 0, \\ Q &= \{x_1 + se \mid (x_1, s) \in X_1 \times R^1, \|x_1\| \leq r_1, 0 \leq s \leq r_2, r_2 > \rho\}, \\ \partial Q &= (B_{r_1} \cap X_1) \cup \partial\{x_1 \oplus se, \|x_1\| \leq r_1, 0 < s \leq r_2\}, \end{aligned}$$

where $e \in X_2$, $\|e\| = 1$. If

$$f|_S \geq \alpha,$$

and

$$f|_{\partial Q} \leq \beta < \alpha.$$

Then $c = \inf_{\phi \in \Gamma} \sup_{x \in Q} f(\phi(x)) \geq \alpha$, if $f(q)$ satisfies $(CPSC)_c$, then c is a critical value for f .

Theorem1.2 Let X be a Banach space and let $f \in C^{1-0}(X, R)$, let $X = X_1 \oplus X_2$ with

$$\dim X_1 < +\infty$$

and

$$\sup_{S_R^1} f < \inf_{X_2} f,$$

where $S_R^1 = \{u \in X_1 \mid \|u\| = R\}$.

Let $B_R^1 = \{u \in X_1, |u| \leq R\}$, $M = \{g \in C(B_R^1, X) \mid g(s) = s, s \in S_R^1\}$

$$c = \inf_{g \in M} \max_{s \in B_R^1} (g(s)).$$

Then $c \geq \inf_{X_2} f$, if f satisfies $(CPSC)_c$ condition, then c is a critical value of f .

In 1978, Rabinowitz [14] firstly used mini-max methods with the classical Palais-Smale condition to study the periodic solutions for second order Hamiltonian systems with the super-quadratic potential:

$$\ddot{q} + V'(q) = 0 \tag{1.1}$$

He proved that

Theorem 1.3([14]) Suppose V satisfies

(V₁) $V \in C^1(R^n, R)$

(V₂) There exist constants $\mu > 2, r_0 > 0$ such that

$$0 < \mu V(x) \leq V'(x) \cdot x, \quad \forall |x| \geq r_0,$$

$$(V_3) \ V(x) \geq 0, \quad \forall x \in R^n,$$

$$(V_4) \ V(x) = o(|x|^2), \text{ as } |x| \rightarrow 0.$$

Then for any $T > 0$, (1.1) has a non-constant T -periodic solution.

In the last 30 years, there were many works for (1.1), we can refer ([3]-[12],[15,17] etc.), and the references there. In this paper, we try to generalize the result of Rabinowitz to local Lipschitz potential, we get the following Theorem:

Theorem 1.4 Suppose V satisfies

$$(V1) \ V \in C^{1-0}(R^n, R);$$

$$(V2) \ \text{There exist constants } \mu_1 > 2, \mu_2 \in R \text{ such that}$$

$$\langle y, x \rangle \geq \mu_1 V(x) + \mu_2, \quad \forall x \in R^n, y \in \partial V(x);$$

$$(V3) \ \text{There are } a_1 > 0, a_2 \in R \text{ such that}$$

$$V(x) \geq a_1 |x|^{\mu_1} + a_2, \quad \forall x \in R^n,$$

$$(V4)$$

$$0 \leq V(x) \leq A|x|^2, |x| \rightarrow 0.$$

Then for any $T < (\frac{2}{A})^{1/2}\pi$, the following system

$$0 \in \ddot{q} + \partial V(q) \tag{1.2}$$

has at least one non-zero T -periodic solution.

For sub-quadratic second order Hamiltonian system, we can get

Theorem 1.5 Suppose V satisfies

$$(V1) \ V \in C^{1-0}(R^n, R);$$

$$(V2') \ \text{There exist constants } \mu_1 < 2, \mu_2 \in R \text{ such that}$$

$$\langle y, x \rangle \leq \mu_1 V(x) + \mu_2, \quad \forall x \in R^n, y \in \partial V(x);$$

$$(V3')$$

$$V(x) \rightarrow +\infty, |x| \rightarrow +\infty;$$

$$(V4')$$

$$V(x) \leq A|x|^2 + a.$$

Then for any $T < (\frac{2}{A})^{1/2}\pi$, (1.2) has at least one T -periodic solution.

2. Some Lemmas

In order to prove Theorem 1.1, we define functional:

$$f(q) = \frac{1}{2} \int_0^T |\dot{q}|^2 dt - \int_0^T V(q) dt, \quad \forall q \in H^1 \tag{2.1}$$

where

$$H^1 = W^{1,2}(R/TZ, R^n). \tag{2.2}$$

Lemma 2.1([6]) Let $\tilde{q} \in H^1$ be such that $\partial f(\tilde{q}) = 0$.
Then $\tilde{q}(t)$ is a T -periodic solution for (1.2).

Lemma 2.2(Sobolev-Rellich-Kondrachov, Compact Imbedding Theorem [1])

$$W^{1,2}(R/TZ, R^n) \subset C(R/TZ, R^n)$$

and the imbedding is compact.

Lemma 2.3(Eberlein-Shmulyan [18]) A Banach space X is reflexive if and only if any bounded sequence in X has a weakly convergent subsequence.

Lemma 2.4([11],[19]) Let $q \in W^{1,2}(R/TZ, R^n)$ and $q(0) = q(T) = 0$

We have Friedrics-Poincare's inequality:

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{\pi}{T}\right)^2 \int_0^T |q(t)|^2 dt.$$

Let $q \in W^{1,2}(R/TZ, R^n)$ and $\int_0^T q(t) dt = 0$, then

(i) We have Poincare-Wirtinger's inequality

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2 dt$$

(ii) We have Sobolev's inequality

$$\max_{0 \leq t \leq T} |q(t)| = \|q\|_\infty \leq \sqrt{\frac{T}{12}} \left(\int_0^T |\dot{q}(t)|^2 dt \right)^{1/2}$$

We define the equivalent norm in $H^1 = W^{1,2}(R/TZ, R^n)$

$$\|q\|_{H^1} = \left(\int_0^T |\dot{q}|^2 dt \right)^{1/2} + |q(0)|$$

Shi Shuzhong[16] generalized the classical Mini-max Theorems including Benci-Rabinowitz's Generalized Mountain-Pass Lemma and Rabinowitz's Saddle Point Theorem to the local Lipschitz functionals with Chang's compactness condition:

Lemma 2.5 Let X be a Banach space, $f \in C^{1-0}(X, R)$. Let $X = X_1 \oplus X_2$, $\dim X_1 < +\infty$, X_2 is closed in X . Let

$$\begin{aligned} B_a &= \{x \in X \mid \|x\| \leq a\}, \\ S &= \partial B_\rho \cap X_2, \rho > 0, \\ Q &= \{x_1 + se \mid (x_1, s) \in X_1 \times R^1, \|x_1\| \leq r_1, 0 \leq s \leq r_2, r_2 > \rho\}, \\ \partial Q &= (B_{r_1} \cap X_1) \cup \partial\{x_1 \oplus se, \|x_1\| \leq r_1, 0 < s \leq r_2\}, \end{aligned}$$

where $e \in X_2$, $\|e\| = 1$. If

$$f|_S \geq \alpha,$$

and

$$f|_{\partial Q} \leq \beta < \alpha,$$

Then $c = \inf_{\phi \in \Gamma} \sup_{x \in Q} f(\phi(x)) \geq \alpha$, if $f(q)$ satisfies $(PSC)_c$, then c is a critical value for f .

Lemma 2.6 Let X be a Banach space and let $f \in C^1(X, R)$, let $X = X_1 \oplus X_2$ with

$$\dim X_1 < +\infty$$

and

$$\sup_{S_R^1} f < \inf_{X_2} f,$$

where $S_R^1 = \{u \in X_1 | |u| = R\}$.

Let $B_R^1 = \{u \in X_1, |u| \leq R\}$, $M = \{g \in C(B_R^1, X) | g(s) = s, s \in S_R^1\}$

$$c = \inf_{g \in M} \max_{s \in B_R^1} (g(s))$$

Then $c \geq \inf_{X_2} f$, if f satisfies $(PSC)_c$ condition, then c is a critical value of f .

Lemma 2.7 Let X be a Banach space, suppose that F defined on X is local Lipschitz functional and lower semi-continuous and bounded from below. Then $\forall \epsilon_n \downarrow 0$, there is a sequence $\{g_n\}$ such that

$$\begin{aligned} F(g_n) &\rightarrow \inf F, \\ (1 + \|g_n\|)F^0(g_n, h) &\geq -\epsilon_n \|h\|. \end{aligned}$$

Proof Applying Ekeland's variational principle ([7,8]), we can get a sequence g_n such that

$$\begin{aligned} F(g_n) &\leq \inf F + \epsilon_n^2, \\ F(g) &\geq F(g_n) - \epsilon_n \delta(g, g_n). \end{aligned}$$

Let $g = g_n + th, t > 0, h \in X$, then we have

$$F(g_n + th) - F(g_n) \geq -\epsilon_n \delta(g_n + th, g_n),$$

where δ is the geodesic distance.

$$F(g_n + th) - F(g_n) \geq -\epsilon_n \int_0^t \frac{\|h\| ds}{1 + \|g_n + sh\|},$$

then

$$\frac{1}{t} F(g_n + th) - F(g_n) \geq -\epsilon_n \frac{1}{t} \int_0^t \frac{\|h\| ds}{1 + \|g_n + sh\|},$$

let $t \rightarrow 0$, we have

$$\begin{aligned} F^0(g_n, h) &\geq \lim_{t \rightarrow 0} \frac{1}{t} (F(g_n + th) - F(g_n)) \\ &\geq -\epsilon_n \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \frac{\|h\| ds}{1 + \|g_n + sh\|} \\ &= -\epsilon_n \|h\| (1 + \|g_n\|)^{-1}. \end{aligned}$$

3. The Proof of Theorems 1.1,1.2,1.4 and 1.5

By Lemma 2.7 and similar arguments of Shi Shuzhong [16], we can prove Theorem 1.1 and 1.2.

Lemma 3.1 If (V1) – (V3) in Theorem 1.4 hold, then $f(q)$ satisfies the (*Cerami – Palais – Smale – Chang*) condition on H^1 .

Proof Let $\{q_n\} \subset H^1$ satisfy

$$f(q_n) \rightarrow c, \quad (1 + \|q_n\|) \min_{x^* \in \partial f(q_n)} \|x^*\| \rightarrow 0, \quad (3.1)$$

Then we claim $\{q_n\}$ is bounded. In fact, by $f(q_n) \rightarrow c$, we have

$$\frac{1}{2} \|\dot{q}_n\|_{L^2}^2 - \int_0^T V(q_n) dt \rightarrow c \quad (3.2)$$

By the definition, we have

$$\langle \partial f(q_n), q_n \rangle = \|\dot{q}_n\|_{L^2}^2 - \int_0^T (\langle \partial V(q_n), q_n \rangle) dt$$

By (V2), for any $v \in \partial V(q_n)$, we have

$$\|\dot{q}_n\|_{L^2}^2 - \int_0^T \langle v, q_n \rangle dt \leq \|\dot{q}_n\|_{L^2}^2 - \int_0^T [\mu_2 + \mu_1 V(q_n)] dt \quad (3.3)$$

By (3.2) and (3.3), $\forall x^* \in \partial f(q_n)$, we have

$$\langle x^*, q_n \rangle \leq a \|\dot{q}_n\|_{L^2}^2 + C_1 + \delta, \quad n \rightarrow +\infty, \quad (3.4)$$

where $C_1 = c\mu_1 - T\mu_2 + \delta$, $\delta > 0$, $a = 1 - \frac{\mu_1}{2} < 0$.

By the above inequality (3.4) and (3.1), we have $\|\dot{q}_n\|_{L^2} \leq M_1$. Then we claim $|q_n(0)|$ is also bounded. Otherwise, there a subsequence, still denoted by q_n , s.t. $|q_n(0)| \rightarrow +\infty$, since $\|\dot{q}_n\| \leq M_1$, then

$$\min_{0 \leq t \leq 1} |q_n(t)| \geq |q_n(0)| - \|\dot{q}_n\|_2 \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty \quad (3.5)$$

We notice that

$$\langle \partial f(q_n), q_n \rangle = \int_0^T [|\dot{q}_n|^2 dt - \langle \partial V(q_n), q_n \rangle] dt \quad (3.6)$$

$$= 2f(q_n) + \int_0^T [2V(q_n) - \langle \partial V(q_n), q_n \rangle] dt \quad (3.7)$$

By (V2) – (V3), $\forall y \in \partial V(x)$ we have

$$\langle y, x \rangle - 2V(x) \geq (\mu_1 - 2)V + \mu_2 \rightarrow +\infty, \quad |x| \rightarrow +\infty$$

By (3.1) and (3.7), we get a contradiction, so $\|q_n\| = \|\dot{q}_n\|_{L^2} + |q_n(0)|$ is bounded.

By the embedding theorem, $\{q_n\}$ has a weakly convergent subsequence which uniformly converges to $q \in H^1$.

Furthermore, by $V \in C^{1-0}$ and the w^* -upper semi-continuity, it's standard step for the rest proof that the weakly convergent subsequence is also strongly convergent to $q \in H^1$.

Now we prove **Theorem 1.4**. In Theorem 1.1, we take

$$X_1 = R^n, X_2 = \{q \in W^{1,2}(R/TZ, R^n), \int_0^T q(t)dt = 0\}$$

$$S = \left\{ q \in X_2 \mid \left(\int_0^T |\dot{q}|^2 dt \right)^{1/2} = \rho > 0 \right\},$$

$$\partial Q = \{x_1 \in R^n \mid |x_1| \leq r_1\} \cup$$

$$\{q = x_1 + se, x_1 \in R^n, e \in X_2, \|e\| = 1, s > 0, \|q\| = (r_1^2 + r_2^2)^{1/2} > \rho\}.$$

When $q \in X_2$, by Sobolev's inequality, $\int_0^T |\dot{q}|^2 dt \rightarrow 0$ implies $\max |q(t)| \rightarrow 0$. So when $\int_0^T |\dot{q}|^2 dt \rightarrow 0$, (V4) implies

$$V(q) \leq A|q|^2$$

When $q \in X_2$, we have Poincare-Wirtinger inequality, so when

$$\rho = \left[\int_0^T |\dot{q}|^2 dt \right]^{1/2} \rightarrow 0$$

We have

$$\begin{aligned} f(q) &\geq \frac{1}{2} \int_0^T |\dot{q}|^2 dt - A \int_0^T |q|^2 dt \\ &\geq \left[\frac{1}{2} - A(2\pi)^{-2} T^2 \right] \rho^2, \end{aligned}$$

On the other hand, if $q \in X_1$, and we take $|x_1| \leq r_1$ very small, then by (V4), we have

$$f(q) = - \int_0^T V(q) dt \leq 0, |q| \rightarrow 0.$$

If

$$q \in \{q = x_1 + se, x_1 \in R^n, e \in X_2, \|e\| = 1, s > 0, \|q\| = (|x_1|^2 + s^2)^{1/2} = R = (r_1^2 + r_2^2)^{1/2} > \rho\},$$

then by (V3) and Jensen's inequality, we have

$$f(q) = \frac{1}{2} s^2 - \int_0^T V(x_1 + se) dt$$

$$\begin{aligned}
& \leq \frac{1}{2}s^2 - \int_0^T (a|x_1 + se|^{\mu_1} + b)dt \\
& \leq \frac{1}{2}s^2 - [aT^{1-\frac{\mu_1}{2}}(\int_0^T |x_1 + se|^2 dt)^{\frac{\mu_1}{2}} + bT] \\
& = \frac{1}{2}s^2 - aT^{1-\frac{\mu_1}{2}}[T|x_1|^2 + s^2 \int_0^T |e(t)|^2 dt]^{\frac{\mu_1}{2}} - bT
\end{aligned}$$

Notice that we can take r_2 large enough, then $(|x_1|^2 + s^2)^{1/2} = R = (r_1^2 + r_2^2)^{1/2}$ is large enough, then $|x_1|$ or s must be large, so $T|x_1|^2 + s^2 \int_0^T |e(t)|^2 dt$ must be large since $\int_0^T |e(t)|^2 > 0$, so that in such case $f(q) < 0$.

The rest of the proof for Theorem 1.4 is obvious.

Using Theorem 1.2 and similar methods for proving Theorem 1.4, we can prove Theorem 1.5, here we omit it

Acknowledgements

The author Zhang Shiqing sincerely thank the supports of NSF of China and the Grant for the Advisors of Ph.D students.

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